

A Sylow theorem for the integral group ring of $\mathrm{PSL}(2, q)$

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Abstract: For $G = \mathrm{PSL}(2, p^f)$ denote by $\mathbb{Z}G$ the integral group ring over G and by $V(\mathbb{Z}G)$ the group of units of augmentation 1 in $\mathbb{Z}G$. Let r be a prime different from p . Using the so called HeLP-method we prove that units of r -power order in $V(\mathbb{Z}G)$ are rationally conjugate to elements of G . As a consequence we prove that subgroups of prime power order in $V(\mathbb{Z}G)$ are rationally conjugate to subgroups of G , if $p = 2$ or $f = 1$.

Let G be a finite group and $\mathbb{Z}G$ the integral group ring over G . Denote by $V(\mathbb{Z}G)$ the group of units of augmentation 1 in $\mathbb{Z}G$, i.e. those units whose coefficients sum up to 1. We say that a finite subgroup U of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup W of G , if there exists a unit $x \in \mathbb{Q}G$ such that $x^{-1}Ux = W$. The question if some, or even all, finite subgroups of $V(\mathbb{Z}G)$ are rationally conjugate to subgroups of G was proposed by H. J. Zassenhaus in the '60s and published in [Zas74]. This so called Zassenhaus Conjectures motivated a lot of research. E.g. A. Weiss proved the strongest version, that all finite subgroups of $V(\mathbb{Z}G)$ are rationally conjugate to subgroups of G , provided G is nilpotent [Wei88] [Wei91]. K. W. Roggenkamp and L. L. Scott obtained a counterexample [Rog91] to this strong conjecture. The version, which asks whether all finite cyclic subgroups of $V(\mathbb{Z}G)$ are rationally conjugate to subgroups of G , the so called First Zassenhaus Conjecture, is however still open, see e.g. [Her08a], [CMdR13]. Though mostly solvable groups were considered when studying such questions, there are some results available for series of non-solvable groups. E.g. a work on the symmetric groups [Pet76] or for Lie-groups of small rank [Ble99]. The groups $\mathrm{PSL}(2, q)$, which are also the object of study in this paper, found also some special attention in [Wag95],

[Her07], [HHK09], [BK11] or in [BM15b].

In this paper we will limit our attention to "Sylow-like" results, i.e. to finite p -subgroups of $V(\mathbb{Z}G)$. We say that a weak Sylow theorem holds for $V(\mathbb{Z}G)$, if every finite p -subgroup of $V(\mathbb{Z}G)$ is isomorphic to some subgroup of G , and that a strong Sylow theorem holds for $V(\mathbb{Z}G)$, if every finite p -subgroup of $V(\mathbb{Z}G)$ is even rationally conjugate to a subgroup of G . First Sylow-like results for integral group rings were obtained in [KR93]. Later M. A. Dokuchaev and S. O. Juriaans proved a strong Sylow theorem for special classes of solvable groups [DJ96] and M. Hertweck, C. Höfert and W. Kimmerle proved a weak Sylow theorem for $\mathrm{PSL}(2, p^f)$, where $p = 2$ or $f \leq 2$. The results of this article are as follows:

Theorem 1: Let $G = \mathrm{PSL}(2, p^f)$, let r be a prime different from p and let u be a torsion unit in $V(\mathbb{Z}G)$ of r -power order. Then u is rationally conjugate to a group element.

Theorem 2: Let $G = \mathrm{PSL}(2, p^f)$ such that $f = 1$ or $p = 2$. Then any subgroup of prime power order of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of G , i.e. a strong Sylow theorem holds in $V(\mathbb{Z}G)$.

1 HeLP-method and known results

Let G be a finite group. A very useful notion to study rational conjugacy of torsion units are partial augmentations: Let $u = \sum_{g \in G} a_g g \in \mathbb{Z}G$ and x^G be the conjugacy class of the element $x \in G$ in G . Then $\varepsilon_x(u) = \sum_{g \in x^G} a_g$ is called the **partial augmentation** of u at x . This relates to rational conjugacy via:

Lemma 1.1 ([MRSW87, Th. 2.5]). *Let $u \in V(\mathbb{Z}G)$ be a torsion unit. Then u is rationally conjugate to a group element if and only if $\varepsilon_x(u^k) \geq 0$ for all $x \in G$ and all powers u^k of u .*

It is well known that if $u \neq 1$ is a torsion unit in $V(\mathbb{Z}G)$, then $\varepsilon_1(u) = 0$ by the so called Berman-Higman Theorem [Seh93, Prop. 1.4]. If $\varepsilon_x(u) \neq 0$, then the order of x divides the order of u [MRSW87, Th. 2.7], [Her06, Prop. 3.1]. Moreover the exponent of G and of $V(\mathbb{Z}G)$ coincide [CL65, Cor. 4.1] and if U is a finite subgroup of $V(\mathbb{Z}G)$ the order of U divides the order of G [ŽK67] (or [Seh93, Lemma 37.3]). We will use these

facts in the following without further mention.

Let u be a torsion unit in $V(\mathbb{Z}G)$ of order n and ζ a primitive complex n -th root of unity and let K be some field, whose characteristic p does not divide n . Let ξ be a (not necessarily primitive) complex n -th root of unity and let D be a K -representation of G with character φ . Here φ is understood as an ordinary or Brauer character. It was first obtained by Luthar and Passi for K having characteristic 0 [LP89] and later generalized by Hertweck for positive characteristic [Her07] that the multiplicity of ξ as an eigenvalue of $D(u)$, which we denote by $\mu(\xi, u, \varphi)$ and which is of course a non-negative integer, may be computed as

$$\mu(\xi, u, \varphi) = \frac{1}{n} \sum_{\substack{d|n \\ d \neq 1}} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\varphi(u^d)\xi^{-d}) + \frac{1}{n} \sum_{\substack{x \in G \\ x \text{ } p\text{-regular}}} \varepsilon_x(u) \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\varphi(x)\xi^{-1}),$$

where as usual $\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(x) = \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(x)$.

The multiplicity of a complex root of unity as an eigenvalue of a matrix over a field of positive characteristic should be understood, here and in the rest of the paper, in the sense of Brauer.

If u is of prime power order r^k for the first sum in the expression above we obtain

$$\frac{1}{n} \sum_{\substack{d|n \\ d \neq 1}} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\varphi(u^d)\xi^{-d}) = \frac{1}{r} \mu(\xi^r, u^r, \varphi).$$

Using these formulas to find possible partial augmentations for torsion units in integral group rings of finite groups is today called HeLP-method. For a diagonalizable matrix A we will write $A \sim (a_1, \dots, a_n)$, if the eigenvalues of A , with multiplicities, are a_1, \dots, a_n .

All subgroups of $G = \text{PSL}(2, p^f)$ were first known to Dickson [Dic01, Theorem 260]. Let $d = \gcd(2, p-1)$. Up to conjugation there is exactly one cyclic group of order p , $\frac{p^f+1}{d}$ and $\frac{p^f-1}{d}$ respectively in G and every element of G lies in a conjugate of such a group. In particular there is only one conjugacy class of involutions in G . The Sylow p -subgroups are elementary-abelian, the Sylow subgroups for all other primes, which are odd, are cyclic and if $p \neq 2$ the Sylow 2-subgroup is dihedral or a Kleinian four-group. There are d conjugacy classes of elements of order p . If $g \in G$ is not of order p or 2 its only distinct conjugate in $\langle g \rangle$ is g^{-1} . We denote by a a fixed element of order $\frac{p^f-1}{d}$ and

by b a fixed element of order $\frac{p^f+1}{d}$.

The modular representation theory of $\text{PSL}(2, p^f)$ in defining characteristic is well known. All irreducible representations were first given by R. Brauer and C. Nesbitt [BN41]. The explicit Brauer table of $\text{SL}(2, p^f)$, which contains the Brauer table of $\text{PSL}(2, p^f)$, may be found in [Sri64]. In particular all characters are real valued since any p -regular element is conjugate to its inverse. However, I was not able to find the following Lemma in the literature, except, without proof, in Hertwecks preprint [Her07], so a short proof is included.

Lemma 1.2. *Let $G = \text{PSL}(2, p^f)$ and $d = \gcd(2, p-1)$. There are p -modular representations of G given by $\Theta_0, \Theta_1, \Theta_2, \dots$ such that there is a primitive $\frac{p^f-1}{d}$ -th root of unity α and a primitive $\frac{p^f+1}{d}$ -th root of unity β satisfying*

$$\begin{aligned}\Theta_k(b) &\sim (1, \beta, \beta^{-1}, \beta^2, \beta^{-2}, \dots, \beta^k, \beta^{-k}), \\ \Theta_k(a) &\sim (1, \alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}, \dots, \alpha^k, \alpha^{-k})\end{aligned}$$

for every $k \in \mathbb{N}_0$.

Proof: The group $\text{SL}(2, p^f)$ acts on the vector space spanned by the homogenous polynomials in two commuting variables x, y of some fixed degree e extending the natural operation on the 2-dimensional vector space spanned by x, y , see e.g. [Alp86, p. 14-16]. Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x^i y^j = (-1)^{i+j} x^i y^j$ this action affords a $\text{PSL}(2, p^f)$ -representation if and only if e is even and p is odd or $p = 2$. So let from now on e be even.

Call this representation $\Theta_{\frac{e}{2}}$. Let γ be an eigenvalue of an element in $\text{SL}(2, q)$ mapping onto a under the natural projection from $\text{SL}(2, p^f)$ to $\text{PSL}(2, p^f)$. Then $\Theta_{\frac{e}{2}}(a)$ has the same eigenvalues as $\Theta_{\frac{e}{2}}\left(\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}\right)$. Now $\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} x^i y^j = \gamma^{i-j} x^i y^j$, so the eigenvalues are $\{\gamma^{i-j} \mid 0 \leq i, j \leq e, i+j=e\} = \{\gamma^{2t} \mid \frac{-e}{2} \leq t \leq \frac{e}{2}\}$. Thus setting $\alpha = \gamma^2$ proves the first part of the claim.

Now let δ be an eigenvalue of an element in $\text{SL}(2, p^f)$ mapping onto b under the natural projection from $\text{SL}(2, p^f)$ to $\text{PSL}(2, p^f)$. The action of $\text{SL}(2, p^f)$ may of course be extended to $\text{SL}(2, p^{2f})$. So $\Theta_{\frac{e}{2}}(b)$ has the same eigenvalues as $\Theta_{\frac{e}{2}}\left(\begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix}\right)$, where the matrix may be seen as an element in $\text{SL}(2, p^{2f})$. Then doing the same calculations as above and setting $\beta = \delta^2$ proves the Lemma.

Notation: The Brauer-character belonging to a representation Θ_i from the Lemma above will be denoted by φ_i .

Using the HeLP-method R. Wagner [Wag95] and Hertweck [Her07] obtained already some results about rational conjugacy of torsion units of prime power order in $\text{PSL}(2, p^f)$. Part of Wagners result was published in [BHK04].

Lemma 1.3. [Wag95] (also [Her07, Prop. 6.1]) *Let $G = \text{PSL}(2, p^f)$ and $f \leq 2$. If u is a unit of order p in $V(\mathbb{Z}G)$, then u is rationally conjugate to a group element.*

Remark: The HeLP-method does not suffice to prove rational conjugacy to a group element of a unit of order p in $V(\mathbb{Z} \text{PSL}(2, p^f))$, if p is odd and $f \geq 3$. There is also no other method or idea around how one could e.g. obtain whether a unit of order 3 in $V(\mathbb{Z} \text{PSL}(2, 27))$ is rationally conjugate to a group element or not.

Lemma 1.4. [Her07, Prop. 6.4] *Let $G = \text{PSL}(2, p^f)$ and let r be a prime different from p . If u is a unit of order r in $V(\mathbb{Z}G)$, then u is rationally conjugate to an element of G .*

Lemma 1.5. [Her07, Prop. 6.5] *Let $G = \text{PSL}(2, p^f)$, let r be a prime different from p and u a torsion unit in $V(\mathbb{Z}G)$ of order r^n . Let $m < n$ and denote by S a set of representatives of conjugacy classes of elements of order r^m in G . Then $\sum_{x \in S} \varepsilon_x(u) = 0$. If moreover g is an element of order r^n in G , then $\mu(1, u, \varphi) = \mu(1, g, \varphi)$ for every p -modular Brauer character φ of G .*

If one is interested not only in cyclic groups the following result is very useful. It may be found e.g. in [Seh93, Lemma 37.6] or in [Val94, Lemma 4].

Lemma 1.6. *Let G be a finite group, U a finite subgroup of $V(\mathbb{Z}G)$ and H a subgroup of G isomorphic to U . If $\sigma : U \rightarrow H$ is an isomorphism such that $\chi(u) = \chi(\sigma(u))$ for all $u \in U$ and all irreducible complex characters χ of G , then U is rationally conjugate to H .*

2 Proof of the results

We will first sum up some elementary number theoretical facts. The notation $a \equiv b \pmod{c}$ will mean, that a is congruent b modulo c .

Proposition 2.1. *Let t and s be natural numbers such that s divides t and denote by ζ_t and ζ_s a primitive complex t -th root of unity and s -th root of unity respectively. Then*

$$\text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\zeta_s) = \mu(s) \frac{\varphi(t)}{\varphi(s)},$$

where μ denotes the Möbius function and φ Euler's totient function. So for a prime r and natural numbers n, m with $m \leq n$ we have

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_{r^n})/\mathbb{Q}}(\zeta_{r^m}) = \begin{cases} r^{n-1}(r-1), & m = 0 \\ -r^{n-1}, & m = 1 \\ 0, & m > 1 \end{cases}$$

Let moreover i and j be integers prime to r , then

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_{r^n})/\mathbb{Q}}(\zeta_{r^m}^i \zeta_{r^m}^{-j}) = \begin{cases} r^{n-1}(r-1), & i \equiv j \pmod{r^m} \\ -r^{n-1}, & i \not\equiv j \pmod{r^m}, \quad i \equiv j \pmod{r^{m-1}} \\ 0, & i \not\equiv j \pmod{r^{m-1}} \end{cases}$$

Proof of Proposition 2.1: Let $s = p_1^{f_1} \cdot \dots \cdot p_k^{f_k}$ be the prime factorisation of s . For a natural number l let $I(l) = \{i \in \mathbb{N} \mid 1 \leq i \leq l, \gcd(i, l) = 1\}$. As is well known, $\mathrm{Gal}(\mathbb{Q}(\zeta_t)/\mathbb{Q}) = \{\sigma_i : \zeta_t \mapsto \zeta_t^i \mid i \in I(t)\}$. From this the case $s = 1$ follows immediately. Otherwise we have

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\zeta_s) = \sum_{i \in I(t)} \zeta_s^i = \frac{\varphi(t)}{\varphi(s)} \sum_{i \in I(s)} \zeta_s^i = \frac{\varphi(t)}{\varphi(s)} \prod_{j=1}^k \sum_{i \in I(p_j^{f_j})} \zeta_{p_j^{f_j}}^i.$$

Now $\sum_{i \in I(p_j^{f_j})} \zeta_{p_j^{f_j}}^i = \begin{cases} -1, & f_j = 1 \\ 0, & f_j > 1 \end{cases}$ and this gives the first formula. The other formulas are special cases of this general formula since $\varphi(r^n) = (r-1)(r^{n-1})$.

Proof of Theorem 1: Let $G = \mathrm{PSL}(2, p^f)$, let r be a prime different from p and let u be a torsion unit in $V(\mathbb{Z}G)$ of order r^n . Let ζ be an primitive complex r^n -th root of unity and set $\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}} = \mathrm{Tr}$. If $n = 1$, then by Lemma 1.4 u is rationally conjugate to an element in G , so assume $n \geq 2$. Assume further that by induction u^r is rationally conjugate to an element in G .

We will proceed by induction on m , where $1 \leq m \leq n$. For a fixed m let $l = \frac{r^m-1}{2}$ if r is odd and $l = \frac{r^m-2}{2}$ if $r = 2$. Let $\{x_i \mid 1 \leq i \leq l, \gcd(i, r) = 1\}$ be a full set of representatives of conjugacy classes of elements of order r^m in G such that $x_1^i = x_i$ (this is possible by the group theoretical properties of G given above) and let S be a set of representatives of conjugacy classes of elements of G of r -power order not greater than r^n containing x_1, \dots, x_l . The proof will be divided in several steps:

a) For $m < n$ we will show $\varepsilon_{x_i}(u) = 0$ for every $1 \leq i \leq l$, i.e. the partial augmentations of u at elements of order r^m vanish. This will be proved by an induction on $k \leq m$ showing

$$(i) \quad \varepsilon_{x_i}(u) = \varepsilon_{x_j}(u) \text{ for } i \equiv \pm j \pmod{r^{m-k}}.$$

If r is even it suffices to prove this for $k = m - 1$, if r is odd we will prove it for $k = m$. It then follows from Lemma 1.5 that $\varepsilon_{x_i}(u) = 0$ for all x_i .

b) For $m = n$ after reordering the x_1, \dots, x_l we will show $\varepsilon_{x_1}(u) = 1$ and $\varepsilon_{x_i}(u) = 0$ for $i \geq 2$. Together with a) this proves the Theorem. This will be achieved by an induction on k to show several facts:

$$(i) \quad \varepsilon_{x_1}(u) = 1 \text{ and } \varepsilon_{x_i}(u) = 0 \text{ for } i \equiv \pm 1 \pmod{r^{n-k}}, i \neq 1.$$

$$(ii) \quad \varepsilon_{x_i}(u) = \varepsilon_{x_j}(u) \text{ for } i \equiv \pm j \pmod{r^{n-k}} \text{ and } i \not\equiv \pm 1 \pmod{r^{n-k}}.$$

These statements will be proved for $k = n - 1$. If r is even this is already enough.

In case r is odd we will moreover prove that $\sum_{i \equiv \alpha(r)} \varepsilon_{x_i}(u) = 0$ for $\alpha \not\equiv \pm 1 \pmod{r}$, which then also implies the Theorem.

So let m be a natural number such that $m < n$. If $m = 0$ statement a) is the Berman-Higman Theorem and if $r = 2$ and $m = 1$ it follows from Lemma 1.5 and the fact that there is only one conjugacy class of involutions in G . So assume we know $\varepsilon_x(u) = 0$ for $\circ(x) < r^m$. The representations and the corresponding characters of G from Lemma 1.2 will be used freely. Statement (i) in a) is certainly true for $k = 0$, so assume $\varepsilon_{x_i}(u) = \varepsilon_{x_j}(u)$ for $i \equiv \pm j \pmod{r^{m-k}}$ for some k . Since u^r is rationally conjugate to a group element, there exists a primitive r^{n-1} -th root of unity $\zeta_{r^{n-1}}$ such that

$$\Theta_{r^k}(u^r) \sim (1, \zeta_{r^{n-1}}, \zeta_{r^{n-1}}^{-1}, \zeta_{r^{n-1}}^2, \zeta_{r^{n-1}}^{-2}, \dots, \zeta_{r^{n-1}}^{r^k}, \zeta_{r^{n-1}}^{-r^k}).$$

Now all p -modular Brauer characters of G are real valued and thus using the last statement in Lemma 1.5 we obtain that $\Theta_{r^k}(u) \sim (1, a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_{r^k}, a_{r^k}^{-1})$, where for every i we have a_i a root of unity such that $a_i^{r^{m-k}} \neq 1$. So for every primitive r^{m-k} -th root of unity $\zeta_{r^{m-k}}$ we have $\mu(\zeta_{r^{m-k}}, u, \varphi_{r^k}) = 0$. Let ζ_{r^m} be a primitive r^m -th root of unity such that we have $\Theta_{r^k}(x_1) \sim (1, \zeta_{r^m}, \zeta_{r^m}^{-1}, \dots, \zeta_{r^m}^{r^k}, \zeta_{r^m}^{-r^k})$ and set $\xi = \zeta_{r^m}^{r^k}$. Let moreover α be a natural number prime to r such that $1 \leq \alpha \leq l$.

Thus $\mu(\xi^\alpha, u, \varphi_{r^k}) = 0$ and $\varepsilon_x(u) = 0$ for $\circ(x) < r^m$. From here on a sum over i , if not specified, will always mean a sum over all i satisfying $1 \leq i \leq l$ and $r \nmid i$. Then using

the HeLP-method we get

$$\begin{aligned}
0 &= \mu(\xi^\alpha, u, \varphi_{r^k}) = \frac{1}{r} \mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) + \frac{1}{r^n} \sum_{x \in S} \varepsilon_x(u) \text{Tr}(\varphi_{r^k}(x) \xi^{-\alpha}) \\
&= \frac{1}{r} \mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) + \frac{1}{r^n} \sum_{\substack{x \in S \\ \circ(x) > r^m}} \varepsilon_x(u) \text{Tr}(\varphi_{r^k}(x) \xi^{-\alpha}) + \frac{1}{r^n} \sum_i \varepsilon_{x_i}(u) \text{Tr}(\varphi_{r^k}(x_i) \xi^{-\alpha}) \\
&= \frac{1}{r} \mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) + \frac{1}{r^n} \sum_{x \in S} \varepsilon_x(u) \text{Tr}(\xi^{-\alpha}) + \frac{1}{r^n} \sum_i \varepsilon_{x_i}(u) \text{Tr}((\xi^i + \xi^{-i}) \xi^{-\alpha}) \\
&= \frac{1}{r} \mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) + \frac{\text{Tr}(\xi^{-\alpha})}{r^n} + \frac{1}{r^n} \sum_i \varepsilon_{x_i}(u) \text{Tr}((\xi^i + \xi^{-i}) \xi^{-\alpha}). \tag{1}
\end{aligned}$$

In the third line we used that if $\tilde{\zeta}$ is a root of unity of r -power order such that $\tilde{\zeta}^{r^{m-k}} \neq 1$, then $\tilde{\zeta}\xi$ has the same order as $\tilde{\zeta}$ and so $\text{Tr}(\tilde{\zeta}\xi) = 0$ by Proposition 2.1. Thus for an element $x \in S$ of order at least r^m we get

$$\text{Tr}(\varphi_{r^k}(x) \xi^{-\alpha}) = \begin{cases} \text{Tr}(\xi^{-\alpha}), & \circ(x) > r^m \\ \text{Tr}(\xi^{-\alpha} + (\xi^i + \xi^{-i}) \xi^{-\alpha}), & x = x_i \end{cases}$$

Note that as i is prime to r the congruence $i \equiv \alpha \pmod{r^{m-k}}$ implies $-i \not\equiv \alpha \pmod{r^{m-k}}$ for $r^{m-k} \notin \{1, 2\}$ and these exceptions don't have to be considered by our assumptions on m and k .

There are now two cases to consider. First assume $k < m - 1$, so ξ is at least of order r^2 and thus $\text{Tr}(\xi) = 0$. Moreover $\mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) = 0$ and using Proposition 2.1 in (1) we obtain

$$\begin{aligned}
0 &= \frac{1}{r^n} \sum_i \varepsilon_{x_i}(u) \text{Tr}((\xi^i + \xi^{-i}) \xi^{-\alpha}) \\
&= \frac{1}{r^n} \sum_{i \equiv \pm \alpha \pmod{r^{m-k}}} \varepsilon_{x_i}(u) (r^{n-1}(r-1)) + \frac{1}{r^n} \sum_{\substack{i \equiv \pm \alpha \pmod{r^{m-k-1}} \\ i \not\equiv \pm \alpha \pmod{r^{m-k}}}} \varepsilon_{x_i}(u) (-r^{n-1}) \\
&= \sum_{i \equiv \pm \alpha \pmod{r^{m-k}}} \varepsilon_{x_i}(u) - \frac{1}{r} \sum_{i \equiv \pm \alpha \pmod{r^{m-k-1}}} \varepsilon_{x_i}(u). \tag{2}
\end{aligned}$$

So

$$r \sum_{i \equiv \pm \alpha \pmod{r^{m-k}}} \varepsilon_{x_i}(u) = \sum_{i \equiv \pm \alpha \pmod{r^{m-k-1}}} \varepsilon_{x_i}(u).$$

But since by induction $\varepsilon_{x_i}(u) = \varepsilon_{x_j}(u)$ for $i \equiv \pm j \pmod{r^{m-k}}$ the summands on the left hand side are all equal and since changing α by r^{m-k-1} does not change the right hand side of the equation we get $\varepsilon_{x_i}(u) = \varepsilon_{x_j}(u)$ for $i \equiv \pm j \pmod{r^{m-k-1}}$.

Now consider $k = m - 1$, then ξ is a primitive r -th root of unity and thus $\text{Tr}(\xi) = -r^{n-1}$ and $\mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) = \mu(1, u^r, \varphi_{r^k}) = 1$. So using Proposition 2.1 in (1) we get

$$\begin{aligned} 0 &= \frac{1}{r} + \frac{-r^{n-1}}{r^n} + \frac{1}{r^n} \sum_{\pm i \not\equiv \alpha(r)} \varepsilon_{x_i}(u)(-2r^{n-1}) + \frac{1}{r^n} \sum_{\pm i \equiv \alpha(r)} \varepsilon_{x_i}(u)(r^{n-1}(r-1) - r^{n-1}) \\ &= \sum_{\pm i \equiv \alpha(r)} \varepsilon_{x_i}(u) - \frac{2}{r} \sum_i \varepsilon_{x_i}(u). \end{aligned} \quad (3)$$

So

$$r \sum_{\pm i \equiv \alpha(r)} \varepsilon_{x_i}(u) = 2 \sum_i \varepsilon_{x_i}(u).$$

Now by Lemma 1.5 the right side of this equation is zero and by induction all summands on the left side are equal. Hence varying α gives $\varepsilon_{x_i}(u) = 0$ for every x_i and part a) is proved.

So assume $m = n$. As in the computation above we have

$$\Theta_{r^k}(u^r) \sim (1, \zeta_{r^{n-1}}, \zeta_{r^{n-1}}^{-1}, \dots, \zeta_{r^{n-1}}^{r^k}, \zeta_{r^{n-1}}^{-r^k})$$

for some primitive r^{n-1} -th root of unity $\zeta_{r^{n-1}}$ and $\Theta_{r^k}(u) \sim (1, a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_{r^k}, a_{r^k}^{-1})$, where a_i are roots of unity such that $a_i^{r^{n-k}} \neq 1$ for $1 \leq i \leq r^k - 1$ and a_{r^k} is some primitive r^{n-k} -th root of unity. Set $\xi = a_{r^k}$ and reorder the x_1, \dots, x_l such that $\Theta_1(x_1) \sim \Theta_1(u)$, but still $x_1^i = x_i$. Then x_1^r is rationally conjugate to u^r . We will proceed by induction on k .

Let α be a natural number prime to r with $1 \leq \alpha \leq l$. Using the HeLP-method and $\varepsilon_x(u) = 0$ for $\circ(x) < r^n$ we obtain, doing the same calculations as in (1):

$$\mu(\xi^\alpha, u, \varphi_{r^k}) = \frac{1}{r} \mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) + \frac{\text{Tr}(\xi^{-\alpha})}{r^n} + \frac{1}{r^n} \sum_i \varepsilon_{x_i}(u) \text{Tr}((\xi^i + \xi^{-i})\xi^{-\alpha}). \quad (4)$$

As u^r is rationally conjugate to x_1^r we know that $\xi^{\pm r}$ are eigenvalues of $\Theta_{r^k}(u^r)$. So we get

$$\mu(\xi^\alpha, u, \varphi_{r^k}) = \begin{cases} 1, & \alpha \equiv \pm 1 \pmod{r^{n-k}} \\ 0, & \text{else} \end{cases} \quad \text{and} \quad \mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) = \begin{cases} 1, & \alpha \equiv \pm 1 \pmod{r^{n-k-1}} \\ 0, & \text{else} \end{cases}$$

There are now several cases to consider: Statement (ii) from b) is clear for $k = 0$. So let $k < n - 1$. For $i \not\equiv \pm 1 \pmod{r^{n-k-1}}$, set $\alpha = i$. Then $\mu(\xi^\alpha, u, \varphi_{r^k}) = \mu(\xi^{r^\alpha}, u^r, \varphi_{r^k}) = 0$ and since ξ has order at least r^2 , also $\text{Tr}(\xi) = 0$. Thus we can do the same computations as in (2) to obtain (ii) for $k + 1$. So (ii) holds for $k = n - 1$.

To obtain the base case for (i) set $k = 0$. Then ξ is at least of order r^2 , i.e. $\text{Tr}(\xi) = 0$, and from (4) we obtain (similar to the computation in (2)):

$$1 = \frac{1}{r} + \varepsilon_{x_1}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 \pmod{r^{n-1}}} \varepsilon_{x_i}(u) \quad (5)$$

and

$$0 = \frac{1}{r} + \varepsilon_{x_\alpha}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 \pmod{r^{n-1}}} \varepsilon_{x_i}(u) \quad (6)$$

for $\alpha \equiv \pm 1 \pmod{r^{n-1}}$ and $\alpha \neq 1$. Subtracting (6) from (5) gives

$$1 = \varepsilon_{x_1}(u) - \varepsilon_{x_\alpha}(u) \quad (7)$$

for every $\alpha \equiv \pm 1 \pmod{r^{n-1}}$ and $\alpha \neq 1$. Let $t = |\{i \in \mathbb{N} | i \leq l, i \equiv \pm 1 \pmod{r^{n-1}}\}|$. Then summing up (5) with the equations (6) for all $\alpha \equiv \pm 1 \pmod{r^{n-1}}$ gives

$$1 = \frac{t}{r} + \sum_{i \equiv \pm 1 \pmod{r^{n-1}}} \varepsilon_{x_i}(u) - \frac{t}{r} \sum_{i \equiv \pm 1 \pmod{r^{n-1}}} \varepsilon_{x_i}(u) = \frac{t}{r} + (1 - \frac{t}{r}) \sum_{i \equiv \pm 1 \pmod{r^{n-1}}} \varepsilon_{x_i}(u).$$

So $\sum_{i \equiv \pm 1 \pmod{r^{n-1}}} \varepsilon_{x_i}(u) = 1$ and the base case of (i) follows from (7).

So assume $0 \leq k < n - 1$. Then ξ is at least of order r^2 , i.e. $\text{Tr}(\xi) = 0$. By induction $\sum_{i \equiv \pm 1 \pmod{r^{n-k}}} \varepsilon_{x_i}(u) = 1$ and for $\alpha \equiv \pm 1 \pmod{r^{n-k}}$ from (4) computing as in (2) we obtain

$$1 = \frac{1}{r} + \sum_{i \equiv \pm 1 \pmod{r^{n-k}}} \varepsilon_{x_i}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 \pmod{r^{n-k-1}}} \varepsilon_{x_i}(u) = \frac{1}{r} + 1 - \frac{1}{r} \sum_{i \equiv \pm 1 \pmod{r^{n-k-1}}} \varepsilon_{x_i}(u).$$

For $\alpha \not\equiv \pm 1 \pmod{r^{n-k}}$ and $\alpha \equiv \pm 1 \pmod{r^{n-k-1}}$ we obtain the same way

$$0 = \frac{1}{r} + \sum_{i \equiv \pm \alpha \pmod{r^{n-k}}} \varepsilon_{x_i}(u) - \frac{1}{r} \sum_{i \equiv \pm 1 \pmod{r^{n-k-1}}} \varepsilon_{x_i}(u).$$

Thus subtracting the last equation from the one before gives

$$1 = 1 - \sum_{i \equiv \pm \alpha(r^{n-k})} \varepsilon_{x_i}(u).$$

The summands on the right hand side are all equal by (ii), so $\varepsilon_{x_\alpha}(u) = 0$, as claimed. Finally let r be odd, $k = n - 1$ and $\alpha \not\equiv \pm 1 \pmod{r}$. Then ξ is of order r , i.e. $\text{Tr}(\xi) = -r^{n-1}$, and $\mu(\xi^{\alpha r}, u^r, \varphi_{r^k}) = \mu(1, u^r, \varphi_{r^k}) = 3$. So from (4) computing as in (5) we obtain

$$0 = \frac{3}{r} + \frac{-r^{n-1}}{r^n} - \frac{2}{r} \sum_i \varepsilon_{x_i}(u) + \sum_{i \equiv \pm \alpha(r)} \varepsilon_{x_i}(u) = \sum_{i \equiv \pm \alpha(r)} \varepsilon_{x_i}(u).$$

As by (ii) all summands in the last sum are equal, we get $\varepsilon_{x_\alpha}(u) = 0$ and the Theorem is finally proved.

Proof of Theorem 2: Let $G = \text{PSL}(2, p^f)$ such that $f = 1$ or $p = 2$. Assume first that r is an odd prime, which is not p , and R is an r -subgroup of $V(\mathbb{Z}G)$. As every r -subgroup of G is cyclic so is R by [Her08b, Theorem A] and thus R is rationally conjugate to a subgroup of G by Theorem 1. If $p \neq 2$ and R is a 2-subgroup of $V(\mathbb{Z}G)$, then R is either cyclic or dihedral or a Kleinian four group by [HHK09, Theorem 2.1]. If R is cyclic, then it is rationally conjugate to a subgroup of G by Theorem 1. If R is dihedral or a Kleinian four group let $S = \langle s \rangle$ be a maximal cyclic subgroup of R . Then any element of R outside of S is an involution and moreover s is rationally conjugate to an element $g \in G$ by Theorem 1. R is isomorphic to some subgroup H of G , such that the maximal cyclic subgroup of H is generated by g . As there is only one conjugacy class of involutions in G every isomorphism σ between R and H mapping s to g satisfies $\chi(\sigma(u)) = \chi(u)$ for every irreducible complex character of G and every $u \in R$. Thus R is rationally conjugate to H by Lemma 1.6.

If $p = 2$ and P is a 2-subgroup of $V(\mathbb{Z}G)$ then all non-trivial elements of P are involutions, so P is elementary abelian and the order of P divides the order of G . As there is again only one conjugacy class of involutions in G every isomorphism σ between P and a subgroup of G isomorphic with P satisfies $\chi(\sigma(u)) = \chi(u)$ for every irreducible complex character of G and every $u \in P$. So P is rationally conjugate to a subgroup of G by Lemma 1.6. Finally if p is odd and P is a p -subgroup of $V(\mathbb{Z}G)$, then P is cyclic of order p and thus rationally conjugate to a subgroup of G by Lemma 1.3.

Remark: Let $G = \mathrm{PSL}(2, p^f)$ and let n be a number prime to p . The structure of the Brauer table of G in defining characteristic yields immediately, that if we can prove that a unit $u \in V(\mathbb{Z}G)$ of order n is rationally conjugate to an element in G applying the HeLP-method to the Brauer table, then these calculations will hold over any $\mathrm{PSL}(2, q)$, if n and q are coprime. In this sense it would be interesting, and seems actually achievable, to determine a subset A_{p^f} of \mathbb{N} such that we can say: The HeLP-method proves that a unit $u \in V(\mathbb{Z}G)$ of order n is rationally conjugate to an element in G if and only if $n \in A_{p^f}$. Test computations yield the conjecture that A_{p^f} actually contains all odd numbers prime to p . If this turned out to be true this would yield, using the results in [Her07], the First Zassenhaus Conjecture for the groups $\mathrm{PSL}(2, p)$, where p is a Fermat- or Mersenne prime.

Other interesting questions concerning torsion units of the integral group ring of $G = \mathrm{PSL}(2, p^f)$ were mentioned at the end of [HHK09] and are still open today: If the order of $u \in V(\mathbb{Z}G)$ is divisible by p , is u of order p ? Are units of order p rationally conjugate to elements of G ? Are p -subgroups in $V(\mathbb{Z}G)$ necessarily abelian? For the last question a positive answer in case $f = 3$ is given in [BM15b].

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